PBW for an inclusion of Lie algebras

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Abstract

Let $\mathfrak{h} \subset \mathfrak{g}$ be an inclusion of Lie algebras with quotient $\mathfrak{h}$-module $\mathfrak{n}$. There is a natural degree filtration on the $\mathfrak{h}$-module $\mathbf{U}(\mathfrak{g})/\mathbf{U}(\mathfrak{g})\mathfrak{h}$ whose associated graded $\mathfrak{h}$-module is isomorphic to $\mathbf{S}(\mathfrak{n})$. We give a necessary and sufficient condition for the existence of a splitting of this filtration. In turn such a splitting yields an isomorphism between the $\mathfrak{h}$-modules $\mathbf{U}(\mathfrak{g})/\mathbf{U}(\mathfrak{g})\mathfrak{h}$ and $\mathbf{S}(\mathfrak{n})$. For the diagonal embedding $\mathfrak{h} \subset \mathfrak{h} \oplus \mathfrak{h}$ the condition is automatically satisfied and we recover the classical Poincaré-Birkhoff-Witt theorem.

The main theorem and its proof are direct translations of results in algebraic geometry, obtained using an *ad hoc* dictionary. This suggests the existence of a unified framework allowing the simultaneous study of Lie algebras and of algebraic varieties, and a closely related work in this direction is on the way.

1. Introduction

1.1. The aim

Let $\mathfrak{h} \hookrightarrow \mathfrak{g}$ be an inclusion of Lie algebras. Denote by $\mathfrak{n}$ the quotient $\mathfrak{g}/\mathfrak{h}$. The quotient $\mathbf{U}(\mathfrak{g})/\mathbf{U}(\mathfrak{g})\mathfrak{h}$ of $\mathbf{U}(\mathfrak{g})$ by the left ideal generated by $\mathfrak{h}$ is naturally an $\mathfrak{h}$-representation. The main purpose of this paper is to answer the following question (the PBW problem):

> *When is $\mathbf{U}(\mathfrak{g})/\mathbf{U}(\mathfrak{g})\mathfrak{h}$ isomorphic to $\mathbf{S}(\mathfrak{n})$ as $\mathfrak{h}$-representations?*

A more precise way of stating the above question is the following. The representation $\mathbf{U}(\mathfrak{g})/\mathbf{U}(\mathfrak{g})\mathfrak{h}$ admits a natural filtration by $\mathfrak{h}$-modules whose associated graded $\mathfrak{h}$-module is $\mathbf{S}(\mathfrak{n})$. We ask for a necessary and sufficient condition for this filtration to split.

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This question is important in deformation quantization, as the space of $\mathfrak{h}$-invariants $(\mathbf{U}(\mathfrak{g})/\mathbf{U}(\mathfrak{g})\mathfrak{h})^{\mathfrak{h}}$ can be given a natural structure of algebra by identifying it with the space of invariant differential operators on a homogeneous space [6]. An open conjecture of Duflo is concerned with understanding the center of this algebra in terms of the Poisson center of $\mathbf{S}(\mathfrak{n})^{\mathfrak{h}}$, which is thought of as the algebra of functions on a Poisson manifold obtained via reduction through the moment map $\mathfrak{g}^{\vee} \to \mathfrak{h}^{\vee}$. In order for this conjecture to make sense one needs to be in a situation where the PBW isomorphism holds. Traditionally this is achieved by assuming that the inclusion $\mathfrak{h} \hookrightarrow \mathfrak{g}$ splits as a map of $\mathfrak{h}$-modules. We will see that this condition is unnecessarily restrictive: there are many pairs of Lie algebras for which there is a PBW isomorphism (and hence it makes sense to study the Duflo problem), but for which the inclusion $\mathfrak{h} \hookrightarrow \mathfrak{g}$ does not split.

1.2. An analogous problem in algebraic geometry

Kontsevich and Kapranov [4] had the insight that we can view the shifted tangent sheaf $\mathsf{T}_\mathsf{Y}[-1]$ of a smooth algebraic variety Y as a Lie algebra object in the derived category $\mathbf{D}(\mathsf{Y})$ of coherent sheaves on Y, with bracket given by the Atiyah class of the tangent sheaf. Moreover, the Atiyah class of any object in $\mathbf{D}(\mathsf{Y})$ gives it the structure of module over this Lie algebra object (see for example [5]). Loosely speaking $\mathbf{D}(\mathsf{Y})$ can be regarded as the category of representations of the shifted tangent sheaf. The role of the trivial representation is played by the structure sheaf $\mathscr{O}_\mathsf{Y}$.

An embedding $\mathfrak{i} : \mathsf{X} \hookrightarrow \mathsf{Y}$ of smooth algebraic varieties can be thought of as giving rise to an inclusion of Lie algebra objects in $\mathbf{D}(\mathsf{X})$

$$\mathfrak{h} = \mathsf{T}_\mathsf{X}[-1] \hookrightarrow \mathfrak{i}^*\mathsf{T}_\mathsf{Y}[-1] = \mathfrak{g}.$$

If E is an object in $\mathbf{D}(\mathsf{Y})$ then the Atiyah class of the restriction $\mathfrak{i}^*\mathsf{E}$ of E to X is precisely the composite of the above inclusion of Lie algebras with the restriction to X of the Atiyah class of E. In other words the functor

$$\mathfrak{i}^* : \mathbf{D}(\mathsf{Y}) \to \mathbf{D}(\mathsf{X})$$

can be interpreted as the restriction functor

$$\mathrm{Res} : \mathfrak{g}\text{-}\mathfrak{Mod} \to \mathfrak{h}\text{-}\mathfrak{Mod}.$$

(We think of all our functors between derived categories as being implicitly derived, so we write $\mathfrak{i}^*$ instead of $\mathbf{L}\mathfrak{i}^*$, etc.)

We now see a dictionary emerging between the worlds of Lie theory and of algebraic geometry. We can use this dictionary to translate naively the PBW question into a problem in algebraic geometry. The following concepts are matched by this dictionary:

Lie theory	Algebraic geometry
Lie algebras $\mathfrak{h}$, $\mathfrak{g}$	varieties X, Y, $\mathfrak{h} = T_X[-1]$, $\mathfrak{g} = T_Y[-1]$
inclusion $\mathfrak{h} \hookrightarrow \mathfrak{g}$	closed embedding $i : X \hookrightarrow Y$
$\mathfrak{h}$-$\mathfrak{Mod}$, $\mathfrak{g}$-$\mathfrak{Mod}$	$\mathbf{D}(X)$, $\mathbf{D}(Y)$
$\mathbf{1}_\mathfrak{h} \in \mathfrak{h}$-$\mathfrak{Mod}$	$\mathscr{O}_X \in \mathbf{D}(X)$
$\mathsf{Res} : \mathfrak{g}$-$\mathfrak{Mod} \to \mathfrak{h}$-$\mathfrak{Mod}$	$i^* : \mathbf{D}(Y) \to \mathbf{D}(X)$
$\mathsf{Ind} : \mathfrak{h}$-$\mathfrak{Mod} \to \mathfrak{g}$-$\mathfrak{Mod}$	$i_! : \mathbf{D}(X) \to \mathbf{D}(Y)$

The last line is motivated by the fact that the induction functor Ind is the left adjoint of the restriction functor, hence in the right column we take the left adjoint $i_!$ of the pull-back functor, which exists for a closed embedding i of smooth varieties.

In representation-theoretic language the $\mathfrak{h}$-representation $\mathbf{U}(\mathfrak{g})/\mathbf{U}(\mathfrak{g})\mathfrak{h}$ arises as

$$\mathbf{U}(\mathfrak{g})/\mathbf{U}(\mathfrak{g})\mathfrak{h} = \mathsf{Res}\,\mathsf{Ind}\,\mathbf{1}_\mathfrak{h} \in \mathfrak{h}\text{-}\mathfrak{Mod}.$$

Using the dictionary the latter corresponds to the object $i^* i_! \mathscr{O}_X$ of the derived category $\mathbf{D}(X)$. Any object E of $\mathbf{D}(X)$ admits a natural filtration by successive truncations $\tau^{\geq k}E$ whose k-th "quotient" is the cohomology sheaf $\mathscr{H}^k(E)[-k]$. An easy local calculation shows that for $E = i^* i_! \mathscr{O}_X$ we have

$$\mathscr{H}^k(i^* i_! \mathscr{O}_X) = \wedge^k N$$

where N is the normal bundle of X in Y. Thus the associated graded object of $i^* i_! \mathscr{O}_X$ is precisely

$$\mathsf{gr}(i^* i_! \mathscr{O}_X) = \bigoplus_k \wedge^k N[-k] = \mathbf{S}(N[-1]).$$

Since $N[-1] = T_Y[-1]|_X / T_X[-1]$ corresponds via the dictionary to $\mathfrak{n} = \mathfrak{g}/\mathfrak{h}$, this is the precise analogue of the statement that $\mathbf{U}(\mathfrak{g})/\mathbf{U}(\mathfrak{g})\mathfrak{h}$ admits a filtration whose associated graded is

$$\mathsf{gr}\,(\mathbf{U}(\mathfrak{g})/\mathbf{U}(\mathfrak{g})\mathfrak{h}) = \mathbf{S}(\mathfrak{n}).$$

The PBW question translates into the following question about a closed embedding $i : X \hookrightarrow Y$.

When is $i^*i_!\mathscr{O}_X$ isomorphic to $\mathbf{S}(N[-1])$ in $\mathbf{D}(X)$?

Just like in the usual PBW problem, this question is better phrased by asking when the above filtration on $i^*i_!\mathscr{O}_X$ splits. This question was addressed and solved recently by D. Arinkin and the second author in [1], where they prove the following result.

Theorem 1.2. *Let $X^{(1)}$ be the first infinitesimal neighborhood of X in Y, $X \hookrightarrow X^{(1)}$. The following are equivalent:*

1. *the truncation filtration on $i^*i_!\mathscr{O}_X$ splits, giving rise to an isomorphism*

$$i^*i_!\mathscr{O}_X \cong \mathbf{S}(N[-1]);$$

2. *the class α is trivial, where*

$$\alpha \in \mathsf{Ext}^1_X(N[-1]^{\otimes 2}, N[-1])$$

   is obtained by composing the class of the normal bundle exact sequence with the Atiyah class of the normal bundle N;
3. *the vector bundle $N[-1]$ admits an extension to $X^{(1)}$.*

It is worth noting that there are many cases where the short exact sequence

$$0 \to T_X \to T_Y|_X \to N \to 0$$

does not split but the obstruction α is nonetheless trivial. For example this is the case when X is any non-linear hypersurface in $Y = \mathbf{P}^n$.

1.3. The result

Our main result is the following translation of the above theorem.

Theorem 1.3. *There exists a Lie algebra $\mathfrak{h}^{(1)}$, containing $\mathfrak{h}$ as a Lie subalgebra, such that the following are equivalent:*

1. *the natural filtration on $\mathbf{U}(\mathfrak{g})/\mathbf{U}(\mathfrak{g})\mathfrak{h}$ splits, giving rise to an isomorphism of $\mathfrak{h}$-modules*
$$\mathbf{U}(\mathfrak{g})/\mathbf{U}(\mathfrak{g})\mathfrak{h} \cong \mathbf{S}(\mathfrak{n});$$
2. *the class α is trivial, where $\alpha \in \mathsf{Ext}^1_\mathfrak{h}(\mathfrak{n}^{\otimes 2}, \mathfrak{n})$ is obtained by composing the class of*
$$0 \longrightarrow \mathfrak{h} \longrightarrow \mathfrak{g} \longrightarrow \mathfrak{n} \longrightarrow 0$$
   with the $\mathfrak{h}$-action;
3. *the $\mathfrak{h}$-representation $\mathfrak{n}$ admits an extension to $\mathfrak{h}^{(1)}$.*

Observe that in the algebro-geometric context $\mathsf{X}^{(1)}$ is singular, even though X and Y are smooth. It turns out that the correct notion of tangent bundle for such singular spaces is that of *tangent complex*, see [3]. Consequently the Lie algebra $\mathfrak{h}^{(1)}$ shall be defined following the analogy with the tangent complex of $\mathsf{X}^{(1)}$, using insight from Koszul duality. The details are presented in Section 2.

The paper is organized as follows. Section 2 is devoted to the definition of the "first order neighborhood Lie algebra" $\mathfrak{h}^{(1)}$ and to the proof that an $\mathfrak{h}$-module E admits an extension to $\mathfrak{h}^{(1)}$ if and only if a certain class $\alpha_\mathsf{E} \in \mathsf{Ext}^1(\mathfrak{n} \otimes \mathsf{E}, \mathsf{E})$ is trivial. In Section 3 we prove a variant of our main theorem for the inclusion $\mathfrak{h} \hookrightarrow \mathfrak{h}^{(1)}$. More precisely, we prove that a natural filtration on $\mathsf{U}(\mathfrak{h}^{(1)})/\mathsf{U}(\mathfrak{h}^{(1)})\mathfrak{h}$ splits if and only if the class $\alpha := \alpha_\mathfrak{n}$ is trivial. The following section is concerned with proving Theorem 1.3. While the main theorem is concerned with the study of the induction-restriction of the trivial representation, we can deduce from this case a general result for any $\mathfrak{h}$-representation. A sketch of the general case is contained in Section 5. We conclude the paper with a very short section in which we give a simple example of a pair of Lie algebras for which the class α is non-trivial. Appendix A contains a proof of the fact that a certain algebra used in the definition of $\mathfrak{h}^{(1)}$ is Koszul.

2. First order neighborhood Lie algebras

Let $\mathfrak{h} \hookrightarrow \mathfrak{g}$ be an inclusion of Lie algebras and denote by $\mathfrak{n}$ the quotient $\mathfrak{h}$-module $\mathfrak{g}/\mathfrak{h}$. In this section we define the obstruction class α and the first order neighborhood Lie algebra $\mathfrak{h}^{(1)}$ that appear in Theorem 1.3. Then we prove that the class α is trivial if and only if $\mathfrak{n}$ admits an extension to $\mathfrak{h}^{(1)}$.

2.1. The extension class α. We begin with the definition of the extension class α that appears in the statement of Theorem 1.3. Consider the short exact sequence of $\mathfrak{h}$-modules

$$0 \to \mathfrak{h} \to \mathfrak{g} \to \mathfrak{n} \to 0. \tag{1}$$

Let E be an $\mathfrak{h}$-module. Tensoring (1) with E yields the sequence

$$0 \to \mathfrak{h} \otimes E \to \mathfrak{g} \otimes E \to \mathfrak{n} \otimes E \to 0 \tag{2}$$

which remains exact because the tensor product of representations is the tensor product of vector spaces endowed with the $\mathfrak{h}$-module structure given by the Leibniz rule. The extension class of (2) is a map $\mathfrak{n} \otimes E \to \mathfrak{h} \otimes E[1]$ in the derived category of $\mathfrak{h}$-representations, which can be post-composed with the action map $\mathfrak{h} \otimes E \to E$ to give the map

$$\alpha_E : \mathfrak{n} \otimes E \to E[1].$$

Equivalently, we can define α_E as the class in $\mathrm{Ext}^1_{\mathfrak{h}}(\mathfrak{n} \otimes E, E)$ corresponding to the bottom extension in the diagram below:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathfrak{h} \otimes E & \longrightarrow & \mathfrak{g} \otimes E & \longrightarrow & \mathfrak{n} \otimes E & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & E & \longrightarrow & Q & \longrightarrow & \mathfrak{n} \otimes E & \longrightarrow & 0.
\end{array}
\tag{3}
$$

Here the $\mathfrak{h}$-module Q is obtained by push-out in the first square of the above diagram. Explicitly, it is given by

$$Q = E \oplus (\mathfrak{g} \otimes E)/\langle (h(x), 0) - (0, h \otimes x) \rangle$$

where for $h \in \mathfrak{h}$ and $x \in E$ we have denoted by $h(x)$ the action of h on x and $h \otimes x$ is viewed as an element of $\mathfrak{g} \otimes E$ via the inclusion of $\mathfrak{h}$ into $\mathfrak{g}$.

We will be particularly interested in the class $\alpha_{\mathfrak{n}}$ of the $\mathfrak{h}$-module $\mathfrak{n}$. This special class will be denoted simply by α.

2.2. The first order neighborhood Lie algebra $\mathfrak{h}^{(1)}$. Consider the Lie algebra $\mathfrak{h}^{(1)}$ defined by

$$\mathfrak{h}^{(1)} := \mathbf{L}(\mathfrak{g})/\langle [h, g] - [h, g]_{\mathfrak{g}} \mid h \in \mathfrak{h}, g \in \mathfrak{g} \rangle$$

where $\mathbf{L}(\mathfrak{g})$ denotes the free Lie algebra generated by the vector space $\mathfrak{g}$ and $\langle \rangle$ stands for "Lie ideal generated by". More precisely $\mathfrak{h}^{(1)}$ is the quotient of $\mathbf{L}(\mathfrak{g})$ in which the bracket between elements of $\mathfrak{h}$ and $\mathfrak{g}$ has been identified

with the original one in $\mathfrak{g}$. Note that to define the Lie algebra $\mathfrak{h}^{(1)}$ we do not need $\mathfrak{g}$ to be a Lie algebra. The precise weaker condition for which this construction makes sense is given in Lemma 2.3 below.

There are natural maps of Lie algebras

$$\mathfrak{h} \hookrightarrow \mathfrak{h}^{(1)} \text{ and } \mathfrak{h}^{(1)} \to \mathfrak{g}$$

which factor the original inclusion $\mathfrak{h} \hookrightarrow \mathfrak{g}$. Given an $\mathfrak{h}$-representation E we can ask whether E extends to a representation of $\mathfrak{h}^{(1)}$. In other words we ask if on the vector space E we can find an $\mathfrak{h}^{(1)}$-module structure whose restriction to $\mathfrak{h}$ via the map $\mathfrak{h} \to \mathfrak{h}^{(1)}$ is the original one. The following lemma shows that this is the case if and only if $\alpha_E = 0$. We state the lemma in a slightly greater generality.

2.3. Lemma. *Let $\mathfrak{h}$ be a Lie algebra and let $\mathfrak{g}$ be an $\mathfrak{h}$-module that contains $\mathfrak{h}$ as an $\mathfrak{h}$-submodule. An $\mathfrak{h}$-module E is the restriction of an $\mathfrak{h}^{(1)}$-module if and only if its class α_E is trivial.*

Proof. We begin with the if part. Assume that the class α_E is trivial. This implies that the sequence (3) splits in the category of $\mathfrak{h}$-modules. Thus we get a map $j : Q \to E$ of $\mathfrak{h}$-modules that splits the canonical map $E \to Q$. Precomposing j with the middle vertical map in (3) yields a map of $\mathfrak{h}$-modules

$$\rho : \mathfrak{g} \otimes E \to E.$$

This map does not define a representation of $\mathfrak{g}$ on E, but it certainly defines a representation of $\mathbf{L}(\mathfrak{g})$ by the universal property of $\mathbf{L}(\mathfrak{g})$. The fact that ρ respects the $\mathfrak{h}$ structure translates into the fact that $\langle [h, g] - [h, g]_{\mathfrak{g}} \rangle$ is in the kernel of this representation. Thus ρ gives an $\mathfrak{h}^{(1)}$-module structure on E which lifts the original $\mathfrak{h}$-module structure because the first square in (3) commutes.

For the only if part assume we have an $\mathfrak{h}^{(1)}$-module structure on E that lifts the $\mathfrak{h}$ structure. Again denote this action by ρ. We can use the explicit description of Q above to define a splitting

$$(x, g \otimes y) \mapsto (x + \rho(g)(y)).$$

This map is obviously a splitting and it respects the $\mathfrak{h}$-module structure because $\langle [h, g] - [h, g]_{\mathfrak{g}} \rangle$ is in the kernel of the representation ρ. □

3. PBW for inclusions into first order neighborhoods

In this section we study the PBW property for the inclusion $j : \mathfrak{h} \hookrightarrow \mathfrak{h}^{(1)}$ of $\mathfrak{h}$ into its first order neighborhood Lie algebra $\mathfrak{h}^{(1)}$. We prove that the PBW theorem holds if and only if the extension class α, defined in Section 2, vanishes.

3.1. We begin with some notation that will be used. Denote the Lie algebra inclusion $\mathfrak{h} \hookrightarrow \mathfrak{g}$ by i. Denote the natural maps of Lie algebras $\mathfrak{h} \to \mathfrak{h}^{(1)}$ and $\mathfrak{h}^{(1)} \to \mathfrak{g}$ by j and k respectively so that $i = k \circ j$. Denote by i^* the restriction functor from $\mathfrak{g}$-modules to $\mathfrak{h}$-modules and by $i_!$ the induction functor in the reverse direction. Thus we have the adjunction $i_! \dashv i^*$. We also have similar functors and adjunctions for the maps j and k. Finally we denote the 1-dimensional trivial representation of the Lie algebra $\mathfrak{h}$ by $\mathbf{1}_\mathfrak{h}$.

3.2. The goal of the current paper is to understand PBW properties for

$$i^*i_!(\mathbf{1}_\mathfrak{h}) = \mathbf{U}(\mathfrak{g}) \otimes_{\mathbf{U}(\mathfrak{h})} \mathbf{1}_\mathfrak{h} = \mathbf{U}(\mathfrak{g})/\mathbf{U}(\mathfrak{g})\mathfrak{h}.$$

In this section we study the object $j^*j_!(\mathbf{1}_\mathfrak{h})$ which is easier to understand. This representation can be described as a quotient of the tensor algebra $\mathbf{T}(\mathfrak{g})$:

$$j^*j_!(\mathbf{1}_\mathfrak{h}) = \mathbf{U}(\mathfrak{h}^{(1)}) \otimes_{\mathbf{U}(\mathfrak{h})} \mathbf{1}_\mathfrak{h} = \mathbf{T}(\mathfrak{g})/\left(J + \mathbf{T}(\mathfrak{g})\mathfrak{h}\right).$$

Here J denotes the two sided ideal generated by $hg - gh - [h, g]_\mathfrak{g}$ for $h \in \mathfrak{h}$ and $g \in \mathfrak{g}$.

3.3. There are two natural increasing filtrations on the $\mathfrak{h}$-module $j^*j_!(\mathbf{1}_\mathfrak{h})$. The first one is induced from the natural filtration on $\mathbf{U}(\mathfrak{h}^{(1)})$, for which elements of $\mathfrak{h}^{(1)}$ have degree 1. The second one is induced by the grading on $\mathbf{T}(\mathfrak{g})$, where elements of $\mathbf{T}^k(\mathfrak{g})$ have degree k. Throughout this paper we shall only work with the latter filtration, which shall be denoted by $F^0 \subset F^1 \subset F^2 \cdots \subset F^k \cdots$. Explicitly, F^k consists of those elements of $j^*j_!(\mathbf{1}_\mathfrak{h})$ that have a lift to $\mathbf{T}(\mathfrak{g})$ of degree $\leq k$.

3.4. Lemma. *The associated graded $\mathfrak{h}$-module $\mathsf{gr}(F^\cdot)$ of the above filtration is precisely $\mathbf{T}(\mathfrak{n})$. In other words the successive quotients F^k/F^{k-1} are isomorphic, as $\mathfrak{h}$-modules, to $\mathfrak{n}^{\otimes k}$.*

Proof. As $j^*j_!\mathbf{1}_\mathfrak{h}$ is a quotient of $\mathbf{T}(\mathfrak{g})$ by the sum of two ideals, we will understand this quotient in two steps corresponding to the two succesive quotients.

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We first give a description of the algebra $A = \mathbf{T}(\mathfrak{g})/J$. The ideal J is generated by the linear subspace $R \subset \mathfrak{g}^{\otimes 2} \oplus \mathfrak{g}$ spanned by elements of the form $hg - gh - [h, g]_{\mathfrak{g}}$ for $h \in \mathfrak{h}, g \in \mathfrak{g}$. Let qR be the image of R through the projection $\mathfrak{g}^{\otimes 2} \oplus \mathfrak{g} \to \mathfrak{g}^{\otimes 2}$, and form the graded algebra $qA = \mathbf{T}(\mathfrak{g})/\langle qR \rangle$, where $\langle qR \rangle$ denotes the two-sided ideal generated by qR. Since qR lies in the kernel of the quotient algebra morphism $\mathbf{T}(\mathfrak{g}) \to \mathrm{gr}(A)$, we obtain a surjective algebra morphism $qA \to \mathrm{gr}(A)$.

The quadratic algebra qA is Koszul (see Appendix A), and for such algebras we can apply a simple criterion [2] to check that the map $qA \to \mathrm{gr}(A)$ is an isomorphism. We describe this result below.

Let $\varphi : qR \to \mathfrak{g}$ be the linear map defined as follows. For $x \in qR$, $\varphi(x)$ is the linear part of a preimage of x under the projection $R \to qR$. This is well defined because $R \cap \mathfrak{g} = 0$. Now Theorem 4.1 in [2] states that the morphism $qA \to \mathrm{gr}(A)$ is an isomorphism if and only if the following conditions are satisfied:

(1) $\mathsf{Im}\,(\varphi \otimes \mathrm{id} - \mathrm{id} \otimes \varphi) \subset qR$ (this map is defined on $qR \otimes \mathfrak{g} \cap \mathfrak{g} \otimes qR$).

(2) $\varphi \circ (\varphi \otimes \mathrm{id} - \mathrm{id} \otimes \varphi) = 0$.

In our situation, qR is the vector subspace of $\mathfrak{g} \otimes \mathfrak{g}$ spanned by $\{hg - gh | h \in \mathfrak{h}, g \in \mathfrak{g}\}$. The map φ maps $hg - gh$ to $[h, g]_{\mathfrak{g}}$. Thus condition (2) follows from the Jacobi identity, while condition (1) is ensured by the stability of $\mathfrak{h}$ under the bracket of $\mathfrak{g}$. We conclude that the map $qA \to \mathrm{gr}(A)$ is an isomorphism.

In particular, the $\mathbf{k}$-vector space A can now be identified with $\mathbf{T}(\mathfrak{n}) \otimes \mathbf{S}(\mathfrak{h})$. Choose $\mathbf{k}$-linear splittings of the projections $\mathfrak{g} \twoheadrightarrow \mathfrak{n}$ and $\mathbf{T}(\mathfrak{h}) \twoheadrightarrow \mathbf{S}(\mathfrak{h})$. Then the composed map

$$\mathbf{T}(\mathfrak{n}) \otimes \mathbf{S}(\mathfrak{h}) \hookrightarrow \mathbf{T}(\mathfrak{g}) \otimes \mathbf{T}(\mathfrak{g}) \to \mathbf{T}(\mathfrak{g}) \twoheadrightarrow A \qquad (4)$$

is an isomorphism of filtered $\mathbf{k}$-vector spaces. Applying the counit of $\mathbf{S}(\mathfrak{h})$ then produces a $\mathbf{k}$-linear projection $\varphi : A \twoheadrightarrow \mathbf{T}(\mathfrak{n})$.

Let us now prove that $\ker(\varphi) = A\mathfrak{h}$. The kernel of φ clearly lies in $A\mathfrak{h}$. Conversely, we now prove that any element $\sum_s a_s h_s \in A\mathfrak{h}$ lies in the kernel. For any s we can write $a_s = \sum_t b_t c_t$ in a unique way with b_t, resp. c_t, in the image of $\mathbf{T}(\mathfrak{n})$, resp. $\mathbf{S}(\mathfrak{h})$, through (4). Then $a_s h_s = \sum_t b_t(c_t h_s) \in \ker(\varphi)$ since $c_t h_s$ lies in the augmentation ideal of $\mathbf{S}(\mathfrak{h})$.

Therefore we get a filtered isomorphism of $\mathbf{k}$-vector spaces $\mathbf{T}(\mathfrak{n}) \xrightarrow{\sim} A/A\mathfrak{h}$ obtained as the composed map

$$\mathbf{T}(\mathfrak{n}) \hookrightarrow A \twoheadrightarrow A/A\mathfrak{h},$$

where the first inclusion is determined by a $\mathbf{k}$-linear splitting of $\mathfrak{g} \twoheadrightarrow \mathfrak{n}$. We now prove that at the level of associated graded it respects $\mathfrak{h}$-module structures on both sides. For any $h \in \mathfrak{h}$ and any monomial $x_1 \cdots x_k \in \mathbf{T}^k(\mathfrak{n})$, the failure of $\mathfrak{h}$-linearity is given by

$$\sum_{i=1}^{k} x_1 \cdots [h, x_i]_{|\mathfrak{h}} \cdots x_k$$

where $[\,,\,]_{|\mathfrak{h}}$ is the $\mathfrak{h}$-part of the bracket, which is defined by means of the above splitting. We conclude with the very simple observation that for any $h \in \mathfrak{h}$ and any $x_1, \ldots, x_k \in \mathfrak{g}$ we have, in F^{k+1},

$$x_1 \cdots h \cdots x_k \in A\mathfrak{h} + F^k.$$

Therefore the failure of $\mathfrak{h}$-linearity vanishes after passing to the associated graded $\mathfrak{h}$-module of $A/A\mathfrak{h}$. [1]. $\qquad\square$

3.5. Next we relate the extension class $\alpha_\mathfrak{n}$ with the filtration $F^{\cdot}$ on $j^* j_! \mathbf{1}_\mathfrak{h}$. The inclusion $F^0 \hookrightarrow F^k$ of the ground field always splits for any $k > 0$. We shall denote the reduced filtration by $\tilde{F}^{\cdot}$.

By the above lemma we have $\tilde{F}^1 \cong \mathfrak{n}$ and $\tilde{F}^2/\tilde{F}^1 \cong \mathfrak{n}^{\otimes 2}$. Hence the inclusion $\tilde{F}^1 \hookrightarrow \tilde{F}^2$ defines a short exact sequence of $\mathfrak{h}$-modules

$$0 \to \mathfrak{n} \to \tilde{F}^2 \to \mathfrak{n}^{\otimes 2} \to 0. \tag{5}$$

The next lemma shows that the extension class of this sequence is precisely the class $\alpha := \alpha_\mathfrak{n} \in \mathsf{Ext}^1_\mathfrak{h}(\mathfrak{n} \otimes \mathfrak{n}, \mathfrak{n})$ defined in (2.1).

3.6. Lemma. *The short exact sequences (3) and (5) are isomorphic and hence both define the same obstruction class α.*

Proof. We construct a map between

$$Q := \mathfrak{n} \oplus (\mathfrak{g} \otimes \mathfrak{n})/\langle (h(x), 0) - (0, h \otimes x)\rangle$$

and $\tilde{F}^2$ which makes all the squares commute. The required map has two components: one from $\mathfrak{n}$ and the other from $\mathfrak{g} \otimes \mathfrak{n}$. The first component is the natural inclusion map $\mathfrak{n} = \tilde{F}^1 \subset \tilde{F}^2$. The second one is given by

$$g \otimes x \mapsto [g \otimes \bar{x}]$$

where we first choose a lift $\bar{x}$ of $x \in \mathfrak{n}$ to $\mathfrak{g}$ and then take the class of $g \otimes \bar{x} \in \mathbf{T}^2\mathfrak{g}$ in $\tilde{F}^2$. A direct computation checkes that the map is well-defined (independent of lifting) and that the resulting map factors through Q. A quick diagram chasing shows that all squares commute. $\qquad\square$

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[1]The very same argument shows that the $\mathfrak{h}$-module isomorphism $\mathbf{T}^k(\mathfrak{n}) \to F^k/F^{k-1}$ constructed this way does not depend on the choice of a $\mathbf{k}$-linear splitting $\mathfrak{g} \twoheadrightarrow \mathfrak{n}$.

3.7. The perhaps surprising result that will be proved in Proposition 3.9 below is that the vanishing of the extension class α, which by the above lemma is only equivalent to the splitting of the first nontrivial inclusion $F^1 \hookrightarrow F^2$, is in fact equivalent to the splitting of the entire filtration $F^\cdot$. We will need the following standard lemma which establishes an isomorphism of $\mathfrak{g}$-modules analogous to the projection formula in algebraic geometry.

3.8. Lemma. *Let $i : \mathfrak{h} \hookrightarrow \mathfrak{g}$ be an inclusion of Lie algebras. Let E be an $\mathfrak{h}$-module and F be a $\mathfrak{g}$-module. Then we have an isomorphism of $\mathfrak{g}$-modules*

$$i_!(E) \otimes F \cong i_!(E \otimes i^*F).$$

Proof. Since the result is well-known to the experts, we only provide a short outline of its proof. Let $\Delta : \mathbf{U}(\mathfrak{g}) \to \mathbf{U}(\mathfrak{g}) \otimes \mathbf{U}(\mathfrak{g})$ be the cocommutative coproduct on the universal enveloping algebra, and let $S : \mathbf{U}(\mathfrak{g}) \to \mathbf{U}(\mathfrak{g})^{\mathsf{op}}$ be the antipode map. We shall freely use the sumless Sweedler notation for the coproduct,

$$\Delta(f) = f^{(1)} \otimes f^{(2)}, \ (\Delta \otimes \mathrm{id}) \circ \Delta(f) = (\mathrm{id} \otimes \Delta) \circ \Delta(f) = f^{(1)} \otimes f^{(2)} \otimes f^{(3)}, \ \ldots$$

It is then straightforward to check that the linear map

$$\varphi : i_!(E) \otimes F = (\mathbf{U}(\mathfrak{g}) \otimes_{\mathbf{U}(\mathfrak{h})} E) \otimes F \to i_!(E \otimes i^*F) = \mathbf{U}(\mathfrak{g}) \otimes_{\mathbf{U}(\mathfrak{h})} (E \otimes F)$$

given by

$$\varphi((f \otimes x) \otimes y) = f^{(1)} \otimes (x \otimes S(f^{(2)})y)$$

is a well-defined isomorphism, with inverse

$$\psi : i_!(E \otimes i^*F) = \mathbf{U}(\mathfrak{g}) \otimes_{\mathbf{U}(\mathfrak{h})} (E \otimes F) \to i_!(E) \otimes F = (\mathbf{U}(\mathfrak{g}) \otimes_{\mathbf{U}(\mathfrak{h})} E) \otimes F$$

given by

$$\psi(f \otimes (x \otimes y)) = (f^{(1)} \otimes x) \otimes f^{(2)}y. \qquad \square$$

3.9. Proposition. *The following two statements are equivalent:*

(a) *The filtration $F^\cdot$ splits.*

(b) *The extension class α is trivial.*

Proof. The implication from (a) to (b) follows from Lemma 3.6. For the other implication we would like to show that the short exact sequences

$$0 \to F^{k-1} \to F^k \to F^k/F^{k-1} = \mathfrak{n}^{\otimes k} \to 0$$

split assuming that the extension class α vanishes. Note that the last equality in the above sequences is proved in Lemma 3.4. Below we will explicitly construct $\mathfrak{h}$-linear maps $\mathfrak{n}^{\otimes k} \to F^k$ that split the above short exact sequences.

By Lemma 2.3 the condition $\alpha = 0$ is equivalent to the existence of a $\mathfrak{h}^{(1)}$-module structure on $\mathfrak{n}$ that extends the $\mathfrak{h}$-module structure on it. Denote by $\overline{\mathfrak{n}}$ such an extension. Note that as a vector space $\overline{\mathfrak{n}}$ is the same as $\mathfrak{n}$. Denote by ad the structure map $\mathfrak{h}^{(1)} \otimes \overline{\mathfrak{n}} \to \overline{\mathfrak{n}}$ for the $\mathfrak{h}^{(1)}$-module $\overline{\mathfrak{n}}$.

We have a natural map of $\mathfrak{h}$-modules $\mathfrak{n} \hookrightarrow j^* j_!(\mathbf{1}_\mathfrak{h})$ for the inclusion of Lie algebras $j : \mathfrak{h} \to \mathfrak{h}^{(1)}$. By adjunction this defines a map of $\mathfrak{h}^{(1)}$-modules

$$j_!(\mathfrak{n}) \to j_!(\mathbf{1}_\mathfrak{h}).$$

Tensoring both sides with $\overline{\mathfrak{n}}$ yields a map $j_!(\mathfrak{n}) \otimes \overline{\mathfrak{n}} \to j_!(\mathbf{1}_\mathfrak{h}) \otimes \overline{\mathfrak{n}}$. Applying the projection formula in Lemma 3.8 (for the inclusion $j : \mathfrak{h} \hookrightarrow \mathfrak{h}^{(1)}$) we get a map

$$j_!(\mathfrak{n}^{\otimes 2}) \to j_!(\mathfrak{n}).$$

Iterating this procedure yields for any nonnegative integer k a map of $\mathfrak{h}^{(1)}$-modules

$$j_!(\mathfrak{n}^{\otimes k+1}) \to j_!(\mathfrak{n}^{\otimes k}).$$

Hence fixing the integer k we can consider the composition

$$j_!(\mathfrak{n}^{\otimes k}) \to j_!(\mathfrak{n}^{\otimes k-1}) \to \cdots \to j_!(\mathbf{1}_\mathfrak{h}).$$

Applying adjunction to this composition we get a map of $\mathfrak{h}$-modules

$$s_k : \mathfrak{n}^{\otimes k} \to j^* j_!(\mathbf{1}_\mathfrak{h}).$$

We need to check that the image of s_k lies inside the k-th step of the filtration $F^{\cdot}$ and that it splits the natural surjective map from F^k to $\mathfrak{n}^{\otimes k}$ constructed in Lemma 3.4.

By construction the maps $t_{k+1} : j_!(\mathfrak{n}^{\otimes k+1}) \to j_!(\mathfrak{n}^{\otimes k})$ fit into the commutative diagram

$$
\begin{array}{ccc}
j_!(\mathfrak{n}^{\otimes k+1}) & \xrightarrow{\ t_{k+1}\ } & j_!(\mathfrak{n}^{\otimes k}) \\
\downarrow{\scriptstyle \psi} & & \uparrow{\scriptstyle \varphi} \\
j_!(\mathfrak{n}^{\otimes k}) \otimes \overline{\mathfrak{n}} & \xrightarrow{\ t_k \otimes \mathsf{id}\ } & j_!(\mathfrak{n}^{\otimes k-1}) \otimes \overline{\mathfrak{n}},
\end{array}
$$

where ψ and φ are the maps defined in the proof of the projection formula Lemma 3.8. This inductive construction begins with the map $t_1 : j_!(\mathfrak{n}) \to j_!(\mathbf{1}_\mathfrak{h})$ which is explicitly given by $f \otimes x \mapsto fx \otimes 1$. Hence with respect to the filtrations induced from $\mathsf{T}(\mathfrak{g})$, t_1 increases the filtration degree by 1.

It is now important to observe that the coproduct used in the definition of φ and ψ not only preserves the filtration for which elements in $\mathfrak{h}^{(1)}$ are of degree 1, but also preserves the filtration induced from $\mathbf{T}(\mathfrak{g})$. This can be seen by observing that the natural surjective map $\mathbf{T}(\mathfrak{g}) \to \mathbf{U}(\mathfrak{h}^{(1)})$ is a morphism of bialgebras (hence in particular a map of coalgebras). Thus by an induction argument we conclude that the maps t_k all increase the filtration degree by 1. The splitting maps s_k can then be described as the compositions

$$\mathfrak{n}^{\otimes k} \xrightarrow{\ \eta_k\ } j^*j_!(\mathfrak{n}^{\otimes k}) \xrightarrow{\ j^*t_k\ } j^*j_!(\mathfrak{n}^{\otimes k-1}) \longrightarrow \cdots \longrightarrow j^*j_!(\mathbf{1}_\mathfrak{h}).$$

Here η_k is the unit of the adjunction applied to $\mathfrak{n}^{\otimes k}$, explicitly given by $x_1 \otimes \cdots \otimes x_k \mapsto 1 \otimes_{\mathbf{U}(\mathfrak{h})} (x_1 \otimes \cdots \otimes x_k)$. As η_k decreases the filtration by k and we have k times t's to post-compose with it, the map s_k will end up being a filtration preserving map, i.e., its image lies inside $F^k \subset j^*j_!(\mathbf{1}_\mathfrak{h})$.

A direct computation of the map s_k shows that it splits the surjective map $F^k \to \mathfrak{n}^{\otimes k}$. For instance in the cases $k = 2$ and $k = 3$, we have

k=2 $\ x_1 \otimes x_2 \mapsto x_1x_2 - \mathsf{ad}(x_1)x_2.$

k=3 $\ x_1 \otimes x_2 \otimes x_3 \mapsto x_1x_2x_3 - x_1\,\mathsf{ad}(x_2)x_3 - x_2\,\mathsf{ad}(x_1)x_3 + \mathsf{ad}(x_2)\,\mathsf{ad}(x_1)x_3 - \mathsf{ad}(x_1)x_2x_3 + \mathsf{ad}(\mathsf{ad}(x_1)x_2)x_3.$

Lifting is assumed in this formula whenever necessary. One can check directly that that this formula is independent of all liftings involved. However it does depend on the choice of the $\mathfrak{h}^{(1)}$-module $\bar{\mathfrak{n}}$ which lifts the $\mathfrak{h}$-module structure on $\mathfrak{n}$. $\qquad\square$

4. PBW for inclusions of Lie algebras

In this section we prove the main result of this paper, Theorem 1.3. Explicitly, we show that the filtration on $\mathbf{U}(\mathfrak{g})/\mathbf{U}(\mathfrak{g})\mathfrak{h}$ splits if and only if the class α vanishes.

4.1. We shall concentrate our attention on the $\mathfrak{h}$-representation

$$i^*i_!(\mathbf{1}_\mathfrak{h}) = \mathbf{U}(\mathfrak{g})/\mathbf{U}(\mathfrak{g})\mathfrak{h}.$$

This module can be realized as the quotient $\mathbf{T}(\mathfrak{g})/(I + \mathbf{T}(\mathfrak{g})\mathfrak{h})$ where I is the two-sided ideal generated by

$$\{g_1 \otimes g_2 - g_2 \otimes g_1 - [g_1, g_2] \mid g_1, g_2 \in \mathfrak{g}\}.$$

We denote by
$$R^0 \subset R^1 \subset \cdots \subset R^k \subset \cdots$$
the filtration by $\mathfrak{h}$-submodules of $\mathfrak{i}^*\mathfrak{i}_!(\mathbf{1}_\mathfrak{h})$ induced from the degree filtration on $\mathbf{T}(\mathfrak{g})$. We set $G^k := R^k/R^{k-1}$.

4.2. Consider the map

$$\mathfrak{j}^*\mathfrak{j}_!(\mathbf{1}_\mathfrak{h}) \to \mathfrak{j}^*k^*k_!\mathfrak{j}_!(\mathbf{1}_\mathfrak{h}) = \mathfrak{i}^*\mathfrak{i}_!(\mathbf{1}_\mathfrak{h})$$

constructed using the unit map of the adjunction $k_! \dashv k^*$. This map preserves the filtrations and descends to maps between associated graded $\mathfrak{h}$-modules
$$\tau : \mathbf{T}(\mathfrak{n}) = \mathrm{gr}(\mathfrak{j}^*\mathfrak{j}_!(\mathbf{1}_\mathfrak{h})) \to \mathrm{gr}(\mathfrak{i}^*\mathfrak{i}_!(\mathbf{1}_\mathfrak{h})).$$

From the descriptions of $\mathfrak{j}^*\mathfrak{j}_!(\mathbf{1}_\mathfrak{h})$ and $\mathfrak{i}^*\mathfrak{i}_!(\mathbf{1}_\mathfrak{h})$ via quotients of $\mathbf{T}(\mathfrak{g})$ we see that the map τ is surjective.

4.3. Lemma. *The kernel of the map τ is generated by the commutators $x \otimes y - y \otimes x$ for $x, y \in \mathfrak{n}$.*

Proof. The idea is to use the classical PBW theorem for a single Lie algebra. Consider the increasing filtration $E^0 \subset \cdots \subset E^k \subset \cdots$ on the universal enveloping algebra $\mathbf{U}(\mathfrak{g})$. The classical PBW theorem asserts that the kernel of the canonical surjective map $\mathfrak{g}^{\otimes k} \to E^k/E^{k-1}$ is generated by the commutators of elements in $\mathfrak{g}$, thus yielding an isomorphism between the k-th symmetric tensors on $\mathfrak{g}$ and E^k/E^{k-1}.

As all these filtrations are compatible (they all arise from the degree filtration on $\mathbf{T}(\mathfrak{g})$), the surjective map $\mathfrak{j}^*\mathfrak{j}_!(\mathbf{1}_\mathfrak{h}) \to \mathfrak{i}^*\mathfrak{i}_!(\mathbf{1}_\mathfrak{h})$ induces surjections on the associated graded to give maps $\mathfrak{n}^{\otimes k} \to G^k$. Consider the following commutative diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & I_1 & \longrightarrow & \mathfrak{g}^{\otimes k} & \longrightarrow & E^k/E^{k-1} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & I_2 & \longrightarrow & \mathfrak{n}^{\otimes k} & \xrightarrow{\tau^k} & G^k & \longrightarrow & 0
\end{array}
$$

where I_1 is the degree k part of the commutator ideal in $\mathbf{T}(\mathfrak{g})$ by the PBW theorem and I_2 is the kernel of the map $\mathfrak{n}^{\otimes k} \to G^k$.

We want to show that I_2 is the k-th commutator in $\mathfrak{n}$. It suffices to show that the map $I_1 \to I_2$ is surjective. By the snake lemma this is equivalent to showing that the map from the kernel of $\mathfrak{g}^{\otimes k} \to \mathfrak{n}^{\otimes k}$ to the kernel of $E^k/E^{k-1} \to G^k$ is surjective.

For that we consider another commutative diagram:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & E^{k-1} & \longrightarrow & E^k & \longrightarrow & E^k/E^{k-1} & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & R^{k-1} & \longrightarrow & R^k & \longrightarrow & G^k & \longrightarrow & 0.
\end{array}
$$

Since all the vertical maps are surjective, the snake lemma shows that we have a surjection from the kernel of $E^k \to R^k$ to the kernel of $E^k/E^{k-1} \to G^k$. But the kernel of $E^k \to R^k$ is the right ideal generated by $\mathfrak{h}$ in $\mathbf{U}(\mathfrak{g})$ intersected with E^k, which is a subset of the kernel of $\mathfrak{g}^{\otimes k} \to \mathfrak{n}^{\otimes k}$. Thus the kernel of $\mathfrak{g}^{\otimes k} \to \mathfrak{n}^{\otimes k}$ also surjects onto the kernel of $E^k/E^{k-1} \to G^k$. Thus the lemma is proved. □

To state an if and only if condition for the PBW property of inclusions of Lie algebras, we need the following lemma concerning the obstruction class α.

4.4. Lemma. *The obstruction class* $\alpha \in \mathsf{Ext}^1(\mathfrak{n} \otimes \mathfrak{n}, \mathfrak{n})$ *factors through* $\mathbf{S}^2(\mathfrak{n})$.

Proof. The lemma can be seen as a corollary of Lemma 3.6 and Lemma 4.3. Indeed, by Lemma 3.6, we can consider the following commutative diagram:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathfrak{n} & \longrightarrow & \tilde{F}^2 & \longrightarrow & \mathfrak{n} \otimes \mathfrak{n} & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathfrak{n} & \longrightarrow & \tilde{R}^2 & \longrightarrow & G^2 & \longrightarrow & 0
\end{array}
$$

where the vertical maps are all defined via the adjunction $\mathfrak{j}^*\mathfrak{j}_!(\mathbf{1}_\mathfrak{h}) \to \mathfrak{i}^*\mathfrak{i}_!(\mathbf{1}_\mathfrak{h})$. By Lemma 4.3 $G^2 = \mathbf{S}^2(\mathfrak{n})$ and the last vertical map is the canonical quotient from the tensor product to the symmetric product. Direct calculation shows that the second square is Cartesian. Thus the lemma is proved. □

We can summarize our main result in the following theorem.

4.5. Theorem. *Let* $\mathbf{k}$ *be a field and let* $\mathfrak{h} \hookrightarrow \mathfrak{g}$ *be an inclusion of Lie algebras over* $\mathbf{k}$*. Consider the two filtrations* $R^0 \subset R^1 \subset \cdots \subset R^k \subset \cdots$ *and* $F^0 \subset F^1 \subset \cdots \subset F^k \subset \cdots$ *defined above. We have:*

– $\mathsf{gr}(F^{\cdot}) = \mathbf{T}(\mathfrak{n})$;

– $\mathsf{gr}(R^{\cdot}) = \mathbf{S}(\mathfrak{n})$.

Moreover, if the field $\mathbf{k}$ *has characteristic zero, then the following are equivalent:*

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(a) *The extension class α is trivial.*

(b) *The filtration $F^0 \subset F^1 \subset \cdots \subset F^k \subset \cdots$ splits;*

(c) *The filtration $R^0 \subset R^1 \subset \cdots \subset R^k \subset \cdots$ splits.*

In fact, if the extension class α is trivial, we have the following explicit splitting of the filtration $R^{\cdot}$ that resembles the standard PBW isomorphism:

$$I : \mathbf{S}(\mathfrak{n}) \to \mathbf{T}(\mathfrak{n}) \cong j^* j_! (\mathbf{1}_{\mathfrak{h}}) \to i^* i_! (\mathbf{1}_{\mathfrak{h}}) \cong \mathbf{U}(\mathfrak{g})/\mathbf{U}(\mathfrak{g})\mathfrak{h}.$$

Here the first arrow is given by any (graded) splitting of the surjective morphism $\mathbf{T}(\mathfrak{n}) \to \mathbf{S}(\mathfrak{n})$ in $\mathfrak{h}$-$\mathfrak{Mod}$.

Proof. The fact that $\mathsf{gr}(F^{\cdot}) = \mathbf{T}(\mathfrak{n})$ is proved in Lemma 3.4, and $\mathsf{gr}(R^{\cdot}) = \mathbf{S}(\mathfrak{n})$ follows from Lemma 4.3. For the second part of the theorem, Proposition 3.9 shows that (a) and (b) are equivalent. By Lemma 4.4 (c) implies (a). Below we will show that (b) implies (c) and hence all of (a), (b), (c) are equivalent.

Assuming a splitting I of the surjection $\mathbf{T}(\mathfrak{n}) \to \mathbf{S}(\mathfrak{n})$ (which always exists over a field of characteristic zero) and a splitting s of the filtration $F^{\cdot}$, we can define the following composition

$$\mathbf{S}^k(\mathfrak{n}) \xrightarrow{I^k} \mathbf{T}^k(\mathfrak{n}) \xrightarrow{s^k} F^k \to R^k.$$

Here the last map is the canonical surjective map from F^k to R^k. The fact that this composition defines a splitting for the filtration $R^{\cdot}$ follows from the following commutative diagram

$$
\begin{array}{ccccc}
\mathbf{S}^k(\mathfrak{n}) & \xrightarrow{\ \ I^k\ \ } & \mathbf{T}^k(\mathfrak{n}) = F^k/F^{k-1} & \xrightarrow{\ \ s^k\ \ } & F^k \\
 & & \downarrow & & \downarrow \\
 & & \mathbf{S}^k(\mathfrak{n}) = R^k/R^{k-1} & \longleftarrow & R^k.
\end{array}
\qquad \square
$$

5. Generalization to any representation

The main goal of this section is to extend Theorem 1.3 from the case of the trivial representation $\mathbf{1}_{\mathfrak{h}}$ to that of an arbitrary finite dimensional $\mathfrak{h}$-representation V. Consider the filtrations $R^0 \subset R^1 \subset \cdots R^k \subset \cdots$, $F^0 \subset F^1 \subset \cdots F^k \subset \cdots$, on $i^* i_!(V)$, $j^* j_!(V)$, respectively, which are induced by the degree filtration on $\mathbf{T}(\mathfrak{g}) \otimes V$. Then we have the following theorem.

5.1. Theorem. *There are isomorphisms of $\mathfrak{h}$-modules $\mathrm{gr}(\mathsf{F}^*) = \mathsf{T}(\mathfrak{n}) \otimes \mathsf{V}$ and $\mathrm{gr}(\mathsf{R}^*) = \mathsf{S}(\mathfrak{n}) \otimes \mathsf{V}$. Moreover, the following are equivalent:*

(a) *The extension classes α and α_V are trivial.*

(b) *The filtration $\mathsf{F}^0 \subset \mathsf{F}^1 \subset \cdots \mathsf{F}^k \subset \cdots$ splits.*

(c) *The filtration $\mathsf{R}^0 \subset \mathsf{R}^1 \subset \cdots \mathsf{R}^k \subset \cdots$ splits.*

Sketch of proof. First observe that as vector spaces we have filtered $\mathbf{k}$-linear isomorphisms

$$i^* i_!(\mathsf{V}) \cong i^* i_!(\mathbf{1}_\mathfrak{h}) \otimes \mathsf{V} \quad \text{and} \quad j^* j_!(\mathsf{V}) \cong j^* j_!(\mathbf{1}_\mathfrak{h}) \otimes \mathsf{V}$$

They are not isomorphisms of $\mathfrak{h}$-modules, but on the level of associated graded they induce $\mathfrak{h}$-module isomorphisms. This proves the first part of the theorem.

Contrary to the situation when the representation is trivial, the inclusions $\mathsf{F}^0 \hookrightarrow \mathsf{F}^k$ and $\mathsf{R}^0 \hookrightarrow \mathsf{R}^k$ do not automatically split. In particular the inclusion $\mathsf{V} = \mathsf{F}^0 = \mathsf{R}^0 \hookrightarrow \mathsf{F}^1 = \mathsf{R}^1$ splits if and only if α_V is trivial.

Finally, if α_V is trivial then there exists an $\mathfrak{h}^{(1)}$-module $\tilde{\mathsf{V}}$ such that $\mathrm{Res}(\tilde{\mathsf{V}}) = \mathsf{V}$. From this we deduce an isomorphism of $\mathfrak{h}$-modules $j^* j_!(\mathsf{V}) \cong j^* j_!(\mathbf{1}_\mathfrak{h}) \otimes \mathsf{V}$. We conclude by using the fact that the Theorem is true for the trivial representation. $\qquad\square$

6. An example of a non trivial class

We now give an example of an inclusion of Lie algebras $\mathfrak{h} \hookrightarrow \mathfrak{g}$ for which the obstruction class is non trivial. Let $\mathfrak{g} = \mathfrak{sl}_2$; recall that it is generated by e, h and f, satisfying the relations

$$[e, f] = h, \quad [h, e] = 2e, \quad [h, f] = -2f.$$

Now let $\mathfrak{h}$ be the Lie subalgebra generated by e and h. Then $\mathfrak{n} = \mathfrak{g}/\mathfrak{h}$ is the one dimensional $\mathfrak{h}$-module generated as a vector space by f, with module structure defined by

$$e \cdot f = 0 \quad \text{and} \quad h \cdot f = -2f.$$

6.1. Proposition. *The obstruction class α is non-trivial.*

Proof. First observe that the Chevalley-Eilenberg 1-cocycle

$$c \in C^1\big(\mathfrak{h}, \mathrm{Hom}(\mathfrak{n}, \mathfrak{h})\big)$$

given by

$$c(e)(f) = e \cdot f - [e, f] = -h, \quad c(h)(f) = h \cdot f - [h, f] = 0$$

is a representative of the exact sequence

$$0 \to \mathfrak{h} \to \mathfrak{g} \to \mathfrak{n} \to 0\,.$$

Therefore the 1-cocycle $a \in C^1\big(\mathfrak{h}, \mathrm{Hom}(\mathfrak{n}^{\otimes 2}, \mathfrak{n})\big)$ given by

$$a(e)(f, f) = -h \cdot f = 2f, \quad a(h)(f, f) = 0$$

is a representative of the obstruction class α.

Finally, observe that since e acts trivially on $\mathfrak{n}$, then it acts trivially on $\mathrm{Hom}(\mathfrak{n}^{\otimes 2}, \mathfrak{n})$. Consequently, for any $b \in \mathrm{Hom}(\mathfrak{n}^{\otimes 2}, \mathfrak{n})$ we have $d(b)(e) = 0$, so that $a \neq d(b)$. Thus $\alpha \neq 0$. $\qquad\square$

A. The algebra *qA* is Koszul

In this appendix we prove that the quadratic algebra qA, defined as the quotient of $\mathbf{T}(\mathfrak{g})$ by the two sided ideal generated by the linear subspace qR of $\mathfrak{g}^{\otimes 2}$ spanned by $\{hg - gh \mid h \in \mathfrak{h}, g \in \mathfrak{g}\}$, is Koszul. We refer to [2] and references therein for the many definitions of Koszulity and their equivalence.

The Koszul complex $K(qA)$ of qA is a subcomplex of the Bar resolution $B_{qA}(\mathbf{k})$ of $\mathbf{k}$ as a left qA-module via the augmentation map $\varepsilon : qA \to \mathbf{k}$:

$$K(qA) := \bigoplus_{i \geq 0} \Big(qA \otimes \tilde{K}^i(qA)\Big)[i] \subset \bigoplus_{i \geq 0} \Big(qA \otimes qA^{\otimes i}\Big)[i] =: B_{qA}(\mathbf{k}),$$

where

$$\tilde{K}^i(qA) := \bigcap_{k=0}^{i-2} \mathfrak{g}^{\otimes k} \otimes qR \otimes \mathfrak{g}^{\otimes i-k-2}.$$

Recall that the differential on the Bar resolution $B_{qA}(\mathbf{k})$ is defined by

$$d(a_0 \otimes \cdots \otimes a_i) = \sum_{k=0}^{i-1} (-1)^k a_0 \otimes \cdots \otimes a_k a_{k+1} \otimes \cdots \otimes a_i + (-1)^i a_0 \otimes \cdots a_{i-1} \varepsilon(a_i).$$

A.1. Proposition. *For any* $i < 0$, $H^i\big(K(qA)\big) = 0$. *In other words the algebra* qA *is Koszul.*

Proof. For any $i > 0$, $\tilde{K}^i(qA) = \big(\wedge^{i-1}\mathfrak{h}\big) \wedge \mathfrak{g}$ is the image of $\mathfrak{h}^{\otimes(i-1)} \otimes \mathfrak{g}$ in $\mathfrak{g}^{\otimes i} \subset qA^{\otimes i}$ through the total antisymmetrization map

$$x_1 \otimes \cdots \otimes x_i \mapsto x_1 \wedge \cdots \wedge x_i := \sum_{\sigma \in S_i} \epsilon(\sigma) x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(i)}.$$

As usual, $\tilde{K}^0(qA) = \mathbf{k}$. Now observe that the only non-zero term in the restriction of the differential of the Bar resolution to $K(qA)$ is the first one:

$$d\big(\sum_s a_0^{(s)} \otimes \cdots \otimes a_i^{(s)}\big) = \sum_s a_0^{(s)} a_1^{(s)} \otimes \cdots \otimes a_i^{(s)}.$$

This is a general fact that is not specific to the peculiar situation we are working at.

In other words, for an element $a \otimes x_1 \wedge \cdots \wedge x_i \in qA \otimes \tilde{K}^i(qA)$ we have

$$d(a \otimes x_1 \wedge \cdots \wedge x_i) = \sum_{j=1}^{i} (-1)^{j-1} a x_j \otimes x_1 \wedge \cdots \wedge \hat{x}_j \wedge \cdots \wedge x_i.$$

Remember that the symmetric algebra $S(\mathfrak{h})$ is Koszul: its Koszul complex $K\big(S(\mathfrak{h})\big)$, which is $\oplus_{i \geq 0} S(\mathfrak{h}) \otimes \wedge^i(\mathfrak{h})[i]$ with differential being given by the formula above, is acyclic in negative degrees. Finally, one sees that $K(qA)$ is isomorphic to the $\mathbf{dg}$ $K\big(S(\mathfrak{h})\big)$-module freely generated by the two step complex

$$\cdots \longrightarrow 0 \longrightarrow \big(T(\mathfrak{n}) \otimes \mathfrak{n}\big)[1] \longrightarrow T(\mathfrak{n}) \longrightarrow 0 \longrightarrow \cdots,$$

which is acyclic in negative degrees. This can be easily seen by considering a $\mathbf{k}$-linear splitting $\mathfrak{n} \hookrightarrow \mathfrak{g}$ of $\mathfrak{g} \twoheadrightarrow \mathfrak{n}$ and observing that

$$K\big(S(\mathfrak{h})\big) \otimes \big(T(\mathfrak{n}) \otimes (\mathfrak{n}[1] \oplus \mathbf{k})\big) \cong qA \otimes \left(\bigoplus_{i \geq 0} \wedge^i(\mathfrak{h})[i]\right) \otimes (\mathfrak{n}[1] \oplus \mathbf{k})$$

$$\cong qA \otimes \left(\mathbf{k} \oplus \bigoplus_{i \geq 0} \big(\wedge^{i-1}(\mathfrak{h})\big) \wedge \mathfrak{g}[i]\right).$$

We leave to the reader the straightforward task of tracking the differential through this identification. $\qquad\square$

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